Shape theory via affine transformation: Some generalisations

José A. Díaz-García *
Department of Statistics and Computation
Universidad Autónoma Agraria Antonio Narro
25350 Buenavista, Saltillo, Coahuila, México
E-mail: jadiaz@uaaan.mx

Francisco J. Caro-Lopera
Department of Basic Sciences
Universidad de Medellín
Carrera 87 No.30-65, of. 5-103
Medellín, Colombia
E-mail: fjcaro@udem.edu.co

Abstract

This work sets the statistical affine shape theory in the context of real normed division algebras. The general densities apply for every field: real, complex, quaternion, octonion, and for any noncentral and non-isotropic elliptical distribution; then the separated published works about real and complex shape distributions can be obtained as corollaries by a suitable selection of the field parameter and univariate integrals involving the generator elliptical function. As a particular case, the complex normal affine density is derived and applied in brain magnetic resonance scans of normal and schizophrenic patients.

1 Introduction

The literature of matrix-variate distributions (real, complex, quaternion, octonion) tells us about a great effort for obtaining separately topics that were noticed recently (Díaz-García (2009), Díaz-García and Gutiérrez-Jáimez (2009), Díaz-García and Gutiérrez-Jáimez (2010)) can be derived under a general approach.

In fact, many models and techniques are explored first in the real case, and then their extensions to the complex case are proposed joint with all the necessary mathematical tools for their development. From the last 60 years we can citate hundreds of examples of these extensions, see Herz (1955) and James (1964), Muirhead (1982) and Khatri (1965), Davis (1980) and Ratnarajah *et al.* (2005), among many others examples.

In the statistical shape theory context, Dryden and Mardia (1998) gives an important summary of diverse techniques in the real case. By other hand, Micheas *et al.* (2006) studied

*Corresponding author
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some of the topics in Dryden and Mardia (1998) for the complex case.

There a number of techniques in shape theory, we focus in this paper in the affine approach. Goodall and Mardia (1993)(see also Díaz-García *et al.* (2003)) proposed an alternative system shape coordinates termed configuration or affine coordinates, randomly indexed by a matrix multivariate gaussian distribution. Then, Caro-Lopera *et al.* (2009) extended this theory by replacing the normality assumption with a matrix multivariate elliptical law. Those works were studied in the real field, so if we follow the tradition of the literature, we can expect an extension to the complex case, for example, by studying the new jacobians, integrals and computations.

Instead of this, we propose an unified approach for the statistical theory of shape, by studying the real, complex, quaternion and octonion cases in a simultaneous way. As we shall see in section 2, these four cases are formally termed, real normed division algebras. However as usual, this type of generalisation have a price, in this case we need some concepts and notation from the abstract algebra.

For the sake of completeness, the case of the octonions is considered, but is important to highlight that many of the results for the octonion field, only can be conjectured, because, many theoretical problems about these numbers remain open, see Dray and Manogue (1999). In fact, the relevance of the octonions for understanding the real world is not clear at present, see Baez (2002).

Section 2 reviews some definitions and notation on real normed division algebras, also, some concepts and integral properties of Jack polynomials and generalised hypergeometric function are given; then in section 3 a Jacobian with respect to Lebesgue measure for real normed division algebras is obtained and the main results follows as a consequence; section 4 gives the affine shape distribution for several particular elliptical laws; and finally, section 5 shows an application from the literature of shape in complex case.

2 Preliminary results

A detailed discussion of real normed division algebras may be found in Baez (2002) and Gross and Richards (1987), and of Jack polynomials and hypergeometric functions in Sawyer (1997), Gross and Richards (1987) and Koev and Edelman (2006). For convenience, we shall introduce some notations, although in general we adhere to standard notations.

For us a **vector space** shall always be a finite-dimensional module over the field of real numbers. An **algebra** $\mathfrak{F}$ shall be a vector space that is equipped with a bilinear map $m : \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}$ termed *multiplication* and a nonzero element $1 \in \mathfrak{F}$ termed the *unit* such that $m(1, a) = m(a, 1) = 1$. As usual, we abbreviate $m(a, b) = ab$ as ab. We do not assume $\mathfrak{F}$ associative. Given an algebra, we shall freely think of real numbers as elements of this algebra via the map $\omega \mapsto \omega 1$.

An algebra $\mathfrak{F}$ is a **division algebra** if given $a, b \in \mathfrak{F}$ with $ab = 0$, then either $a = 0$ or $b = 0$. Equivalently, $\mathfrak{F}$ is a division algebra if the operation of left and right multiplications by any nonzero element are invertible. A **normed division algebra** is an algebra $\mathfrak{F}$ that is also a normed vector space with $||ab|| = ||a||||b||$. This implies that $\mathfrak{F}$ is a division algebra and that $||1|| = 1$.

There are exactly four real finite-dimensional normed division algebras: real numbers, complex numbers, quaternions and octonions, these being denoted generically as $\mathfrak{F}$, see Baez (2002). All division algebras have a real dimension of $1, 2, 4$ or 8, respectively, whose dimension is denoted by β, see Baez (2002, Theorems 1, 2 and 3).

Let $\mathcal{L}^{\beta}_{m,n}$ be the linear space of all $n \times m$ matrices of rank $m \leq n$ over $\mathfrak{F}$ with m distinct positive singular values, where $\mathfrak{F}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{F}^{n \times m}$ be the set of all $n \times m$ matrices over $\mathfrak{F}$. The dimension of $\mathfrak{F}^{n \times m}$ over $\Re$ is βmn.

Let $\mathbf{A} \in \mathfrak{F}^{n\times m}$, then $\mathbf{A}^* = \overline{\mathbf{A}}^T$ denotes the usual conjugate transpose.

The set of matrices $\mathbf{H}_1 \in \mathfrak{F}^{n\times m}$ such that $\mathbf{H}_1^*\mathbf{H}_1 = \mathbf{I}_m$ is a manifold denoted $\mathcal{V}_{m,n}^\beta$, termed *Stiefel manifold* ($\mathbf{H}_1$ is also known as *semi-orthogonal* ($\beta = 1$), *semi-unitary* ($\beta = 2$), *semi-symplectic* ($\beta = 4$) and *semi-exceptional type* ($\beta = 8$) matrices, see Dray and Manogue (1999)). The dimension of $\mathcal{V}_{m,n}^\beta$ over $\Re$ is $[\beta mn - m(m-1)\beta/2 - m]$. In particular, $\mathcal{V}_{m,m}^\beta$ with dimension over $\Re$, $[m(m+1)\beta/2 - m]$, is the maximal compact subgroup $\mathfrak{U}^\beta(m)$ of $\mathcal{L}_{m,m}^\beta$ and consist of all matrices $\mathbf{H} \in \mathfrak{F}^{m\times m}$ such that $\mathbf{H}^*\mathbf{H} = \mathbf{I}_m$. Therefore, $\mathfrak{U}^\beta(m)$ is the *real orthogonal group* $\mathcal{O}(m)$ ($\beta = 1$), the *unitary group* $\mathcal{U}(m)$ ($\beta = 2$), *compact symplectic group* $\mathcal{S}p(m)$ ($\beta = 4$) or *exceptional type matrices* $\mathcal{O}o(m)$ ($\beta = 8$), for $\mathfrak{F} = \Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$, respectively.

Denote by $\mathfrak{S}_m^\beta$ the real vector space of all $\mathbf{S} \in \mathfrak{F}^{m\times m}$ such that $\mathbf{S} = \mathbf{S}^*$. Let $\mathfrak{P}_m^\beta$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{F}^{m\times m}$; then $\mathfrak{P}_m^\beta$ is an open subset of $\mathfrak{S}_m^\beta$. Over $\Re$, $\mathfrak{S}_m^\beta$ consist of *symmetric* matrices; over $\mathfrak{C}$, *Hermitian* matrices; over $\mathfrak{H}$, *quaternionic Hermitian* matrices (also termed *self-dual matrices*) and over $\mathfrak{O}$, *octonionic Hermitian* matrices. Generically, the elements of $\mathfrak{S}_m^\beta$ are termed as **Hermitian matrices**, irrespective of the nature of $\mathfrak{F}$. The dimension of $\mathfrak{S}_m^\beta$ over $\Re$ is $[m(m-1)\beta + 2]/2$.
Let $\mathfrak{D}_m^\beta$ be the *diagonal subgroup* of $\mathcal{L}_{m,m}^\beta$ consisting of all $\mathbf{D} \in \mathfrak{F}^{m\times m}$, $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_m)$.

For any matrix $\mathbf{X} \in \mathfrak{F}^{n\times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{F}^{m\times n}$, $\mathfrak{S}_m^\beta$, $\mathfrak{D}_m^\beta$ or $\mathcal{V}_{m,n}^\beta$, see Dimitriu (2002).

If $\mathbf{X} \in \mathfrak{F}^{n\times m}$ then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{F}^{n\times m}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m} dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$

If $\mathbf{S} \in \mathfrak{S}_m^\beta$ (or $\mathbf{S} \in \mathfrak{T}_L^\beta(m)$) then $(d\mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}_m^\beta$ or in $\mathfrak{T}_L^\beta(m)$) denotes the exterior product of the $m(m+1)\beta/2$ functionally independent variables (or denotes the exterior product of the $m(m-1)\beta/2 + n$ functionally independent variables, if $s_{ii} \in \Re$ for all $i = 1, \dots, m$)

$$(d\mathbf{S}) = \begin{cases} \displaystyle\bigwedge_{i\le j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}, & \\ \displaystyle\bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}, & \text{if } s_{ii} \in \Re. \end{cases}$$

Generally the context establishes the conditions on the elements of $\mathbf{S}$, that is, if $s_{ij} \in \Re$, $\in \mathfrak{C}$, $\in \mathfrak{H}$ or $\in \mathfrak{O}$. It is considered that

$$(d\mathbf{S}) = \bigwedge_{i\le j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)} \equiv \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

Note that, the Lebesgue measure $(d\mathbf{S})$ requires that $\mathbf{S} \in \mathfrak{P}_m^\beta$, that is, $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $\mathbf{\Lambda} \in \mathfrak{D}_m^\beta$ then $(d\mathbf{\Lambda})$ (the Legesgue measure in $\mathfrak{D}_m^\beta$) denotes the exterior product of the βm functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^{n}\bigwedge_{k=1}^{\beta} d\lambda_i^{(k)}.$$

If $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$ then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{m} \mathbf{h}_j^* d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}_1|\mathbf{H}_2) = (\mathbf{h}_1, \dots, \mathbf{h}_m|\mathbf{h}_{m+1}, \dots, \mathbf{h}_n) \in \mathfrak{U}^\beta(m)$. It can be proved that this differential form does not depend on the choice of the matrix $\mathbf{H}_2$ When $m = 1$; $\mathcal{V}_{1,n}^\beta$ defines the unit sphere in $\mathfrak{F}^n$. This is, of course, an $(n-1)\beta$- dimensional surface in $\mathfrak{F}^n$. When $m = n$ and denoting $\mathbf{H}_1$ by $\mathbf{H}$, $(\mathbf{H}^* d\mathbf{H})$ is termed the *Haar measure* on $\mathfrak{U}^\beta(m)$.

The surface area or volume of the Stiefel manifold $\mathcal{V}_{m,n}^\beta$ is

$$\mathrm{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} (\mathbf{H}_1^* d\mathbf{H}_1) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^\beta[n\beta/2]}, \qquad (1)$$

where $\Gamma_m^\beta[a]$ denotes the multivariate Gamma function for the space $\mathfrak{S}_m^\beta$, and is defined by

$$\begin{aligned} \Gamma_m^\beta[a] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \mathrm{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} (d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a - (i-1)\beta/2], \end{aligned}$$

where $\mathrm{etr}(\cdot) = \exp(\mathrm{tr}(\cdot))$, $|\cdot|$ denotes the determinant and $\mathrm{Re}(a) > (m-1)\beta/2$, see Gross and Richards (1987).

Let $C_\kappa^\beta(\mathbf{B})$ be the Jack polynomials of $\mathbf{B} = \mathbf{B}^*$, corresponding to the partition $\kappa = (k_1, \dots k_m)$ of k, $k_1 \geq \cdots \geq k_m \geq 0$ with $\sum_{i=1}^m k_i = k$, see Sawyer (1997) and Koev and Edelman (2006). In addition,

$${}_pF_q^\beta(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{B}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a_1]_\kappa^\beta, \dots, [a_p]_\kappa^\beta}{[b_1]_\kappa^\beta, \dots, [b_q]_\kappa}^\beta \frac{C_\kappa^\beta(\mathbf{B})}{k!},$$

defines the hypergeometric function with one matrix argument on the space of hermitian matrices, where $[a]_\kappa^\beta$ denotes the generalised Pochhammer symbol of weight κ, defined as

$$[a]_\kappa^\beta = \prod_{i=1}^{m} (a - (i-1)\beta/2)_{k_1}$$

where $\Re(a) > (m-1)\beta/2 - k_m$ and $(a)_i = a(a+1)\cdots(a+i-1)$, see Gross and Richards (1987), Koev and Edelman (2006) and Díaz-García (2009).

Lemma 2.1. *If* $\mathbf{X} \in \mathfrak{L}_{m,n}^\beta$, *then*

$$\int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} (\mathrm{tr}(\mathbf{X}\mathbf{H}_1))^{2k} (d\mathbf{H}_1) = \sum_{\kappa} \frac{\left(\frac{1}{2}\right)_k}{[\beta n/2]_\kappa^\beta} C_\kappa^\beta(\mathbf{X}\mathbf{X}^*), \qquad (2)$$

See Díaz-García and Gutiérrez-Jáimez (2010).

Now, we use the complexification $\mathfrak{S}_m^{\beta,\mathfrak{C}} = \mathfrak{S}_m^\beta + i\mathfrak{S}_m^\beta$ of $\mathfrak{S}_m^\beta$. That is, $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consists of all matrices $\mathbf{X} \in (\mathfrak{F}^\mathfrak{C})^{m \times m}$ of the form $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$, with $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_m^\beta$. We refer to $\mathbf{X} = \mathrm{Re}(\mathbf{Z})$ and $\mathbf{Y} = \mathrm{Im}(\mathbf{Z})$ as the *real and imaginary parts* of $\mathbf{Z}$, respectively. The *generalised right half-plane* $\mathbf{\Phi} = \mathfrak{P}_m^\beta + i\mathfrak{S}_m^\beta$ in $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consists of all $\mathbf{Z} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ such that $\mathrm{Re}(\mathbf{Z}) \in \mathfrak{P}_m^\beta$, see Gross and Richards (1987, p. 801).

The next result generalises one given in Xu and Fang (1989), Teng *et al.* (1989) and Caro-Lopera *et al.* (2009) for the real normed division algebras, see Díaz-García (2009):

Lemma 2.2. *Let* $\mathbf{Z} \in \mathbf{\Phi}$ *and* $\mathbf{U} \in \mathfrak{S}_m^\beta$. *Then,*

$$\int_{\mathbf{X} \in \mathfrak{P}_m^\beta} h(\operatorname{tr} \mathbf{XZ}) |\mathbf{X}|^{a-(m-1)\beta/2-1} C_\kappa^\beta(\mathbf{XU})(d\mathbf{X}) = \frac{[a]_\kappa^\beta \Gamma_m^\beta[a]}{\Gamma[am+k]} |\mathbf{Z}|^{-a} C_\kappa^\beta(\mathbf{UZ}^{-1})\gamma, \quad (3)$$

where

$$\gamma = \int_{z \in \mathfrak{P}_1^\beta} h(z) z^{am+k-1} dw < \infty, \quad (4)$$

for $\operatorname{Re}(a) > (m-1)\beta/2 - k_m$, $\kappa = (k_1, \dots k_m)$ *a partition of* k, $k_1 \geq \cdots \geq k_m \geq 0$ *with* $\sum_{i=1}^m k_i = k$.

3 Affine shape distribution

Start with the following definition extended from Goodall and Mardia (1993).

Definition 3.1. Two figures $\mathbf{X} \in \mathcal{L}_{K,N}^\beta$ and $\mathbf{X}_1 \in \mathcal{L}_{K,N}^\beta$ have the *same configuration*, or *affine shape*, if $\mathbf{X}_1 = \mathbf{XE} + \mathbf{1}_N e^*$, for some translation $e \in \mathcal{L}_{1,N}^\beta$ and a $\mathbf{E} \in \mathcal{L}_{K,K}^\beta$.

The configuration coordinates are constructed in the two steps summarised in the expression

$$\mathbf{LX} = \mathbf{Y} = \mathbf{UE}. \quad (5)$$

The matrix $\mathbf{U} \in \mathcal{L}_{K,N-1}^\beta$ contains configuration coordinates of $\mathbf{X}$. Let $\mathbf{Y}_1 \in \mathcal{L}_{K,K}^\beta$ and $\mathbf{Y}_2 \in \mathcal{L}_{K,q}^\beta$, with $q = N - K - 1 \geq 1$, such that $\mathbf{Y} = (\mathbf{Y}_1^* \mid \mathbf{Y}_2^*)^*$. Define also $\mathbf{U} = (\mathbf{I} \mid \mathbf{V}^*)^*$, then $\mathbf{V} = \mathbf{Y}_2\mathbf{Y}_1^{-1}$ and $\mathbf{E} = \mathbf{Y}_1$ where $\mathbf{L} \in \mathcal{L}_{N,N-1}^\beta$ is a Helmert sub-matrix.

Consider the following extension of Caro-Lopera *et al.* (2009, Lemma 8) for real normed division algebras.

Lemma 3.1. *Let* $(\mathbf{F}^{1/2})^2 = \mathbf{F} \in \mathfrak{P}_K^\beta$, $\mathbf{H} \in \mathfrak{U}^\beta(K)$, *and let* $\mathbf{E} = \mathbf{F}^{1/2}\mathbf{H}$ *such that* $\mathbf{Y} = \mathbf{UF}^{1/2}\mathbf{H}$. *Then*

$$(d\mathbf{Y}) = 2^{-K} |\mathbf{F}|^{(q-1)\beta/2} (d\mathbf{V})(d\mathbf{F})(\mathbf{H}^* d\mathbf{H}).$$

Proof. Let $\mathbf{E} = \mathbf{F}^{1/2}\mathbf{H}$, where $\mathbf{E} \in \mathcal{L}_{K,K}^\beta$, $\mathbf{H} \in \mathfrak{U}^\beta(K)$ and $\mathbf{F}^{1/2} \in \mathfrak{P}_K^\beta$. Therefore

$$\mathbf{E}^*\mathbf{E} = \mathbf{H}^*\mathbf{FH}.$$

Then from Díaz-García and Gutiérrez-Jáimez (2009, Lemma 2.2)

$$(d(\mathbf{E}^*\mathbf{E})) = |\mathbf{H}^*\mathbf{H}|^{(K-1)\beta/2+1}(d\mathbf{F}) = |\mathbf{H}|^{(K-1)\beta+2}(d\mathbf{F}) = (d\mathbf{F}).$$

Now, by Díaz-García and Gutiérrez-Jáimez (2009, Lemma 2.15)

$$\begin{aligned}(d\mathbf{E}) &= 2^{-K} |\mathbf{E}^*\mathbf{E}|^{\beta/2-1} (d(\mathbf{E}^*\mathbf{E}))(\mathbf{H}^* d\mathbf{H}) \\ &= 2^{-K} |\mathbf{H}^*\mathbf{FH}|^{\beta/2-1} (d\mathbf{F})(\mathbf{H}^* d\mathbf{H}) \\ &= 2^{-K} |\mathbf{F}|^{\beta/2-1} (d\mathbf{F})(\mathbf{H}^* d\mathbf{H}) \quad (6)\end{aligned}$$

By other hand, observe that

$$\mathbf{Y} = \begin{pmatrix} \mathbf{I} \\ \mathbf{V} \end{pmatrix} \mathbf{E} = \begin{pmatrix} \mathbf{E} \\ \mathbf{VE} \end{pmatrix}.$$

And by differentiating and computing the exterior product, see Díaz-García and Gutiérrez-Jáimez (2009, Lemma 2.1), we obtain

$$(d\mathbf{Y}) = |\mathbf{E}|^{\beta q} (d\mathbf{V})(d\mathbf{E}),$$

but observing that $|\mathbf{E}| = |\mathbf{F}^{1/2}\mathbf{H}| = |\mathbf{F}|^{1/2}$, we get

$$(d\mathbf{Y}) = |\mathbf{F}|^{\beta q/2}(d\mathbf{V})(d\mathbf{E}). \tag{7}$$

By replacing (6) into (7) the desired result is obtained. $\square$ $\square$

Now, recall that $\mathbf{X} \in \mathfrak{L}^{\beta}_{m,n}$ has a matrix multivariate elliptically contoured distribution for real normed division algebras if its density, with respect to the Lebesgue measure, is given by (see Díaz-García and Gutiérrez-Jáimez (2009)):

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{1}{|\mathbf{\Sigma}|^{\beta n/2}|\mathbf{\Theta}|^{\beta m/2}} h\left\{\mathrm{tr}\left[\mathbf{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})^{*}\mathbf{\Theta}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right]\right\},$$

where $\boldsymbol{\mu} \in \mathfrak{L}^{\beta}_{m,n}$, $\mathbf{\Sigma} \in \mathfrak{P}^{\beta}_{m}$, $\mathbf{\Theta} \in \mathfrak{P}^{\beta}_{m}$. The function $h : \mathfrak{F} \to [0,\infty)$ is termed the generator function, and it is such that $\int_{\mathfrak{P}^{\beta}_{1}} u^{\beta nm-1}h(u^2)du < \infty$.

Such a distribution is denoted by $\mathbf{X} \sim \mathcal{E}^{\beta}_{n\times m}(\boldsymbol{\mu}, \mathbf{\Sigma}\otimes\mathbf{\Theta}, h)$, for real case see Fang and Zhang (1990) and Gupta, and Varga (1993) and Micheas *et al.* (2006) for complex case. Observe that this class of matrix multivariate distributions includes gaussian, contaminated normal, Pearson type II and VI, Kotz, Jensen-Logistic, power exponential, Bessel, among other distributions; whose distributions have tails that are weighted more or less, and/or distributions with greater or smaller degree of kurtosis than the gaussian distribution.

Now, we have the mathematical and statistics tools for establishing the main density.

Theorem 3.1. *Let* $\mathbf{X} \sim \mathcal{E}^{\beta}_{N-1\times K}(\boldsymbol{\mu}_{\mathbf{X}}, \mathbf{\Sigma}_{\mathbf{X}}\otimes\mathbf{\Theta}, h)$. *Then the affine shape density is given by*

$$\frac{\pi^{\beta K^2/2}\Gamma^{\beta}_{K}\left[\beta(N-1)/2\right]}{\Gamma^{\beta}_{K}\left[\beta K/2\right]|\mathbf{\Sigma}|^{\beta K/2}|\mathbf{U}^{*}\mathbf{\Sigma}^{-1}\mathbf{U}|^{\beta(N-1)/2}}\sum_{t=0}^{\infty}\frac{1}{t!\Gamma\left[K(N-1)/2+t\right]}\sum_{r=0}^{\infty}\frac{[\mathrm{tr}\,\mathbf{\Omega}]^{r}}{r!}$$
$$\times\sum_{\tau}\frac{[\beta(N-1)/2]^{\beta}_{\tau}}{[\beta K/2]^{\beta}_{\tau}}C^{\beta}_{\tau}(\mathbf{U}^{*}\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}^{*}\mathbf{\Sigma}^{-1}\mathbf{U})^{-1})\gamma, \tag{8}$$

where

$$\gamma = \int_{z\in\mathfrak{P}^{\beta}_{1}} h^{(2t+r)}(z)z^{\beta K(N-1)2+t-1}dz < \infty, \tag{9}$$

and $\mathbf{\Sigma} = \mathbf{L}\mathbf{\Sigma}\mathbf{L}^{*}$, $\boldsymbol{\mu} = \boldsymbol{L}\boldsymbol{\mu}_{\mathbf{X}}$ *and* $\mathbf{\Omega} = \mathbf{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{\Theta}\boldsymbol{\mu}^{*}$.

Proof. Define

$$\mathbf{L}\mathbf{X}\mathbf{\Theta}^{-1/2} = \mathbf{L}\mathbf{Z} = \mathbf{Y} = \mathbf{U}\mathbf{E},$$

where $\left(\mathbf{\Theta}^{1/2}\right)^2 = \mathbf{\Theta}$. Thus $\mathbf{Y} \sim \mathcal{E}^{\beta}_{N-1\times K}(\boldsymbol{\mu}\mathbf{\Theta}^{-1/2}, \mathbf{\Sigma}\otimes\mathbf{I}, h)$ where $\mathbf{\Sigma} = \mathbf{L}\mathbf{\Sigma}\mathbf{L}^{*}$, $\boldsymbol{\mu} = \boldsymbol{L}\boldsymbol{\mu}_{\mathbf{X}}$. Therefore the density of $\mathbf{Y}$ is given by

$$\frac{1}{|\mathbf{\Sigma}|^{\beta K/2}}h\{\mathrm{tr}[\mathbf{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu}\mathbf{\Theta}^{-1/2})(\mathbf{Y}-\boldsymbol{\mu}\mathbf{\Theta}^{-1/2})^{*}]\}.$$

Making the factorisation $\mathbf{Y} = \mathbf{U}\mathbf{E} = \mathbf{U}\mathbf{F}^{1/2}\mathbf{H}$ and by Lemma 3.1, then the joint density of $\mathbf{U}$, $\mathbf{F}$ and $\mathbf{H}$ is

$$\frac{|\mathbf{F}|^{\beta(q+1)/2-1}}{2^{K}|\mathbf{\Sigma}|^{\beta K/2}}h\left[\mathrm{tr}\left(\mathbf{F}\mathbf{U}^{*}\mathbf{\Sigma}^{-1}\mathbf{U}+\mathbf{\Omega}\right)+\mathrm{tr}\left(-2\mathbf{\Theta}^{-1/2}\boldsymbol{\mu}^{*}\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}^{1/2}\mathbf{H}\right)\right]$$
$$\times(\mathbf{H}^{*}d\mathbf{H})(d\mathbf{F})(d\mathbf{V}),$$

where $\mathbf{\Omega} = \mathbf{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{\Theta}^{-1}\boldsymbol{\mu}^{*}$.

Expanding h in series of power, the joint density of $\mathbf{U}$, $\mathbf{F}$ and $\mathbf{H}$ becomes:

$$\frac{|\mathbf{F}|^{\beta(q+1)/2-1}}{2^K|\mathbf{\Sigma}|^{\beta K/2}}\sum_{t=0}^{\infty}\frac{1}{t!}h^{(t)}\left[\operatorname{tr}\left(\mathbf{F}\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U}+\mathbf{\Omega}\right)\right]\left[\operatorname{tr}\left(-2\mathbf{\Theta}^{-1/2}\boldsymbol{\mu}^*\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}^{1/2}\mathbf{H}\right)\right]^t$$
$$(\mathbf{H}^*d\mathbf{H})(d\mathbf{F})(d\mathbf{V}).$$

Now, using the Lemma 2.1 for to integrate with respect to $\mathbf{H}$, and recalling that $(1/2)_t 4^t/(2t)! = 1/t!$ and $C_\tau^\beta(a\mathbf{B}) = a^t C_\tau^\beta(\mathbf{B})$, the marginal joint density of $\mathbf{F}$ and $\mathbf{U}$ is

$$\frac{\pi^{\beta K^2/2}|\mathbf{F}|^{\beta(q+1)/2-1}}{|\mathbf{\Sigma}|^{\beta K/2}\Gamma_K^\beta\left[\beta K/2\right]}\sum_{t=0}^{\infty}\frac{1}{t!}h^{(2t)}\left[\operatorname{tr}\left(\mathbf{F}\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U}+\mathbf{\Omega}\right)\right]$$
$$\times\sum_{\tau}\frac{1}{[\beta K/2]_\tau^\beta}C_\tau^\beta\left(\mathbf{U}^*\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}\right)(d\mathbf{F})(d\mathbf{V}),$$

Assuming that $h^{2t}(\cdot)$ can be expanding in series of power, then the joint density of $\mathbf{F}$ and $\mathbf{U}$ is

$$\frac{\pi^{\beta K^2/2}|\mathbf{F}|^{\beta(q+1)/2-1}}{|\mathbf{\Sigma}|^{\beta K/2}\Gamma_K^\beta\left[\beta K/2\right]}\sum_{t=0}^{\infty}\frac{1}{t!}\sum_{r=0}^{\infty}\frac{1}{r!}h^{(2t+r)}\left[\operatorname{tr}\left(\mathbf{F}\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U}\right)\right]\left[\operatorname{tr}\left(\boldsymbol{\mu}^*\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)\right]^r$$
$$\times\sum_{\tau}\frac{1}{[\beta K/2]_\tau^\beta}C_\tau^\beta\left(\mathbf{U}^*\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}\right)(d\mathbf{F})(d\mathbf{V}).$$

Hence, the marginal density of $\mathbf{U}$ is

$$\frac{\pi^{\beta K^2/2}}{|\mathbf{\Sigma}|^{\beta K/2}\Gamma_K^\beta\left[\beta K/2\right]}\sum_{t=0}^{\infty}\frac{1}{t!}\sum_{r=0}^{\infty}\frac{1}{r!}\left[\operatorname{tr}\left(\boldsymbol{\mu}^*\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)\right]^r\sum_{\tau}\frac{1}{[\beta K/2]_\tau^\beta}$$
$$\times\int_{\mathbf{F}>\mathbf{0}}h^{(2t+r)}\left(\operatorname{tr}\left(\mathbf{F}\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U}\right)\right)|\mathbf{F}|^{\beta(q+1)/2-1}C_\tau^\beta\left(\mathbf{U}^*\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}\right)(d\mathbf{F}). \tag{10}$$

From (3), the integral in (10) is evaluated as

$$\int_{\mathbf{F}>\mathbf{0}}h^{(2t+r)}\left[\operatorname{tr}\left(\mathbf{F}\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U}\right)\right]|\mathbf{F}|^{\beta(q+1)/2-1}C_\tau^\beta\left(\mathbf{U}^*\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}\mathbf{F}\right)(d\mathbf{F})$$
$$=\frac{[\beta(N-1)/2]_\tau\,\Gamma_K^\beta\left[\beta(N-1)/2\right]}{\Gamma\left[K(N-1)/2+t\right]}\frac{C_\tau^\beta(\mathbf{U}^*\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U})^{-1})}{|\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U}|^{\beta(N-1)/2}}\gamma,$$

where

$$\gamma=\int_{z\in\mathfrak{P}_1^\beta}h^{(2t+r)}(z)z^{\beta K(N-1)/2+t-1}dz<\infty,$$

and the required result follows. □ □

Note that the general density is indexed by a simple univariate integral involving the general derivative of generator function. These kind of densities appear rare in matrix-variate distributions (see Caro-Lopera *et al.* (2009)). However, they demand the computation of derivatives of any order, which is not a trivial fact; general formulae for the classical elliptical models (Kotz, Pearson, Bessel, Jensen-Logistic) are available too in the above mentioned reference.

Finally, the central and isotropic affine shape densities are obtained.

Corollary 3.1. *If* $\mathbf{X} \sim \mathcal{E}^{\beta}_{N-1\times K}(\mathbf{0}, \mathbf{\Sigma_X} \otimes \mathbf{\Theta}, h)$*, then the central affine shape density is invariant under the elliptical distributions, moreover, its density is*

$$\frac{\Gamma^{\beta}_{K}\left[\beta(N-1)/2\right]}{\pi^{\beta Kq/2}\Gamma^{\beta}_{K}\left[\beta K/2\right]|\mathbf{\Sigma}|^{\beta K/2}}|\mathbf{U}^{*}\mathbf{\Sigma}^{-1}\mathbf{U}|^{-\beta(N-1)/2}.$$

Proof. The proof follows by taking $t = r = 0$ in (8) and noting that $h^{(2t+r)}(z) = h^{(0)}(z) \equiv h(z)$, hence

$$\frac{\pi^{\beta K^2/2}\Gamma^{\beta}_{K}\left[\beta(N-1)/2\right]}{|\mathbf{\Sigma}|^{\beta K/2}\Gamma^{\beta}_{K}\left[\beta K/2\right]}\frac{|\mathbf{U}^{*}\mathbf{\Sigma}^{-1}\mathbf{U}|^{-\beta(N-1)/2}}{\Gamma\left[\beta K(N-1)/2\right]}\gamma.$$

Now, using Fang and Zhang (1990, p. 59),

$$\gamma = \int_{0}^{\infty} h(z)z^{\beta K(N-1)/2-1}dz = \frac{\Gamma\left[\beta K(N-1)/2\right]}{\pi^{\beta K(N-1)/2}},$$

the desired result is obtained. □ □

Corollary 3.2. *If* $\mathbf{Y} \sim \mathcal{E}^{\beta}_{N-1\times K}(\boldsymbol{\mu}_{\mathbf{X}}, \sigma^2\mathbf{I}_{N-1} \otimes \mathbf{\Theta}, h)$*, then the isotropic noncentral affine shape density is given by*

$$\frac{\pi^{\beta K^2/2}\Gamma^{\beta}_{K}\left[\beta(N-1)/2)\right]}{\Gamma^{\beta}_{K}\left[\beta K/2\right]|\mathbf{I}_K + \mathbf{V}^{*}\mathbf{V}|^{\beta(N-1)/2}}\sum_{t=0}^{\infty}\frac{1}{t!\Gamma\left[K(N-1)/2+t\right]}\sum_{r=0}^{\infty}\frac{1}{r!}\left[\mathrm{tr}\left(\mathbf{\Omega}_1\right)\right]^{r}$$
$$\times\sum_{\tau}\frac{\left[\beta(N-1)/2\right]^{\beta}_{\tau}}{\left[\beta K/2\right]^{\beta}_{\tau}}C^{\beta}_{\tau}\left(\mathbf{U}^{*}\mathbf{\Omega}_1\mathbf{U}(\mathbf{I}_K+\mathbf{V}^{*}\mathbf{V})^{-1}\right)\gamma,$$

where

$$\gamma = \int_{z\in\mathfrak{P}^{\beta}_{1}} h^{(2t+r)}(z)z^{\beta K(N-1)/2+t-1}dz < \infty,$$

and $\mathbf{\Omega}_1 = \sigma^{-2}\boldsymbol{\mu}\mathbf{\Theta}^{-1}\boldsymbol{\mu}^{*}$.

Proof. The result follows easily, just recall that $\mathbf{U} = (\mathbf{I}_K \mid \mathbf{V}^{*})^{*}$ and take $\mathbf{\Sigma} = \sigma^2\mathbf{I}_{N-1}$ in (8). Thus

$$|\mathbf{\Sigma}|^{\beta K/2}|\mathbf{U}^{*}\mathbf{\Sigma}^{-1}\mathbf{U}|^{\beta(N-1)/2} = |\mathbf{I}_K + \mathbf{V}^{*}\mathbf{V}|^{\beta(N-1)/2}. \quad \square$$

□

4 The Gaussian affine shape distribution

In this section we study the Gaussian affine shape distribution, as corollary of the preceding results.

Corollary 4.1. *Let* $\mathbf{X} \sim \mathcal{E}^{\beta}_{N-1\times K}(\boldsymbol{\mu}_{\mathbf{X}}, \mathbf{\Sigma_X} \otimes \mathbf{\Theta}, h)$*. Then the affine shape density is given by*

$$\frac{\Gamma^{\beta}_{K}\left[\beta(N-1)/2\right]\mathrm{etr}\{-\beta(\mathbf{\Omega} - \mathbf{U}^{*}\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}^{*}\mathbf{\Sigma}^{-1}\mathbf{U})^{-1})/2\}}{\pi^{\beta Kq/2}\Gamma^{\beta}_{K}\left[\beta K/2\right]|\mathbf{\Sigma}|^{\beta K/2}|\mathbf{U}^{*}\mathbf{\Sigma}^{-1}\mathbf{U}|^{\beta(N-1)/2}}$$
$${}_1F^{\beta}_{1}(-\beta q/2; \beta K/2; -\beta\mathbf{U}^{*}\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}^{*}\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}/2). \tag{11}$$

Proof. From Díaz-García and Gutiérrez-Jáimez (2009), the gaussian case turns,

$$h(v)=\frac{1}{\left(\frac{2\pi}{\beta}\right)^{\beta K(N-1)/2}}\exp\{-\beta v/2\}$$

and

$$h^{2t+r}(v)=\frac{\left(-\frac{\beta}{2}\right)^{2t+r}}{\left(\frac{2\pi}{\beta}\right)^{\beta K(N-1)/2}}\exp\{-\beta v/2\}.$$

Therefore

$$\begin{aligned}\gamma &= \int_{z\in\mathfrak{P}_1^\beta} h^{(2t+r)}(z)z^{\beta K(N-1)/2+t-1}dz\\ &= \int_{z\in\mathfrak{P}_1^\beta}\frac{\left(-\frac{\beta}{2}\right)^{2t+r}}{\left(\frac{2\pi}{\beta}\right)^{\beta K(N-1)/2}}\exp\{-\beta z/2\}dz\\ &= \frac{\Gamma[\beta K(N-1)/2+t]}{\pi^{\beta K(N-1)/2}}\left(-\frac{\beta}{2}\right)^r\left(\frac{\beta}{2}\right)^t.\end{aligned}$$

Also, note that

$$[\mathrm{tr}\,\mathbf{\Omega}]^r\left(-\frac{\beta}{2}\right)^r=[\mathrm{tr}\,-\beta\mathbf{\Omega}/2]^r$$

and

$$\left(\frac{\beta}{2}\right)^t C_\tau^\beta(\mathbf{U}^*\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U})^{-1})=C_\tau^\beta(\beta\mathbf{U}^*\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}/2).$$

The final expression for (11) is obtained by applying the Kummer relations, see Díaz-García (2009). □ □

The isotropic case of this distribution is given by

Corollary 4.2. *Let* $\mathbf{X}\sim\mathcal{E}^\beta_{N-1\times K}(\boldsymbol{\mu}_{\mathbf{X}},\mathbf{\Sigma}_{\mathbf{X}}\otimes\mathbf{\Theta},h)$. *Then the affine shape density is given by*

$$\begin{aligned}&\frac{\Gamma_K^\beta[\beta(N-1)/2]\,\mathrm{etr}\{-\beta(\mathbf{\Omega}-\mathbf{U}^*\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U})^{-1})/2\}}{\pi^{\beta Kq/2}\Gamma_K^\beta[\beta K/2]\,|\mathbf{I}_K+\mathbf{V}^*\mathbf{V}|^{\beta(N-1)/2}}\\ &\qquad {}_1F_1^\beta(-\beta q/2;\beta K/2;-\beta\mathbf{U}^*\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{U}(\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U})^{-1}/2) \qquad (12)\\ &=\frac{\Gamma_K^\beta[\beta(N-1)/2]\,\mathrm{etr}\{-\beta(\sigma^{-2}\boldsymbol{\mu}\boldsymbol{\mu}^*-\sigma^{-2}\mathbf{U}^*\boldsymbol{\mu}\boldsymbol{\mu}^*\mathbf{U}(\mathbf{U}^*\mathbf{U})^{-1})/2\}}{\pi^{\beta Kq/2}\Gamma_K^\beta[\beta K/2]\,|\mathbf{I}_K+\mathbf{V}^*\mathbf{V}|^{\beta(N-1)/2}}\\ &\qquad {}_1F_1^\beta(-\beta q/2;\beta K/2;-\beta\sigma^{-2}\mathbf{U}^*\boldsymbol{\mu}\boldsymbol{\mu}^*\mathbf{U}(\mathbf{U}^*\mathbf{U})^{-1}/2). \qquad (13)\end{aligned}$$

It follows straightforwardly, recall that $\mathbf{U}=(\mathbf{I}_K\mid\mathbf{V}^*)^*$ and replace $\mathbf{\Sigma}=\sigma^2\mathbf{I}_{N-1}$ in (12). Thus

$$|\mathbf{\Sigma}|^{\beta K/2}|\mathbf{U}^*\mathbf{\Sigma}^{-1}\mathbf{U}|^{\beta(N-1)/2}=|\mathbf{I}_K+\mathbf{V}^*\mathbf{V}|^{\beta(N-1)/2}.\quad\square$$

5 Example

In this last section we apply the complex normal shape model in a classical data, the brain magnetic resonance scans of normal and schizophrenic patients (see Bookstein (1966), Dryden and Mardia (1998), among many others). The random samples consist of 14 scans of each group, which correspond to a near midsagittal two dimensional slices of MR. On each image, an expert locates the following 13 landmarks and registers their coordinates. The selected points correspond to (Bookstein (1966), Dryden and Mardia (1998)): 1. splenium, 2. genu, 3. top of corpus callosum, 4. top of head, 5. tentorium of cerebellum at dura, 6. top of cerebellum, 7. tip of fourth ventricle, 8. bottom of cerebellum, 9. top of pons, 10. bottom of pons, 11. optic chiasm, 12. frontal pole, 13. superior colliculus (see Dryden and Mardia (1998, figure 9, p.12)). The aim of the experiment is to test the equality in mean shape of the two population after removing the non geometrical information of the scans.

The preceding works have analysed the shape under Euclidian transformation by filtering out translation, scaling and rotation of scans, and concluding that both mean shapes are statistically different. However, the shape theory via similarity transformation works well when the objects are rigid and develop a"constant radial growth" in some sense, but in the case of the brain, which is a soft organ, it is prone to deformations, so in order to match the scans we need to filter out the shear, instead of the rotation. Then the affine shape or configuration analysis is more appropriate than the usual Euclidian shape (this discrepancy can be solved statistically by using the so termed modified BIC criteria, see Yang and Yang (2007) and the references therein).

Now, if the original landmark 13×1 vector $\mathbf{X}$ follows an isotropic complex normal model, then from corollary 4.2 we have that the corresponding affine shape density is given by

$$\frac{\Gamma_K^\beta[\beta(N-1)/2]\operatorname{etr}\{-\beta\sigma^{-2}\boldsymbol{\mu}\boldsymbol{\mu}^*/2+\beta\sigma^{-2}\mathbf{U}^*\boldsymbol{\mu}\boldsymbol{\mu}^*\mathbf{U}(\mathbf{U}^*\mathbf{U})^{-1}/2\}}{\pi^{\beta Kq/2}\Gamma_K^\beta[\beta K/2]\,|\mathbf{U}^*\mathbf{U}|^{\beta(N-1)/2}}$$
$$\quad {}_1F_1^\beta(-\beta q/2;\beta K/2;-\beta\sigma^{-2}\mathbf{U}^*\boldsymbol{\mu}\boldsymbol{\mu}^*\mathbf{U}(\mathbf{U}^*\mathbf{U})^{-1}/2). \tag{14}$$

where $\beta=2$, $K=1$, $N=3$, $q=11$. Then it is of interest the estimation of the scale parameter σ^2 and the location parameter

$$\boldsymbol{\mu}=(\mu_{1,1}+i\mu_{1,2},\mu_{2,1}+i\mu_{2,2},\mu_{3,1}+i\mu_{3,2},\mu_{4,1}+i\mu_{4,2},\mu_{5,1}+i\mu_{5,2},\mu_{6,1}+i\mu_{6,2},$$
$$\mu_{7,1}+i\mu_{7,2},\mu_{8,1}+i\mu_{8,2},\mu_{9,1}+i\mu_{9,2},\mu_{10,1}+i\mu_{10,2},\mu_{11,1}+i\mu_{11,2},\mu_{12,1}+i\mu_{12,2})'$$

Note that the above affine density is a polynomial of degree eleven, then the inference procedure is notoriously simple.

Let $\mathfrak{L}(\widetilde{\boldsymbol{\mu}},\widetilde{\sigma}^2,h)$ be the log likelihood function of a given group-model. The maximisation of the likelihood function $\mathfrak{L}(\widetilde{\boldsymbol{\mu}},\widetilde{\sigma}^2,h)$, is obtained in this paper by using the *Nelder-Mead Simplex Method*, which is an unconstrained multivariable function using a derivative-free method; specifically, we apply the routine **fminsearch** implemented by the sofware MatLab.

The shape densities are polynomials of scalar zonal polynomials, i.e. hypergeometric series which terminates and this can be computed easily by the algorithms of Koev and Edelman (2006).

At this point the log likelihood can be computed, then we use fminsearch for the MLE's. The initial value for the algorithm is the sample mean of the normal variables $\mathbf{Y}\sim\mathcal{N}_{12\times 1}(\boldsymbol{\mu},\sigma^2\mathbf{I}_{12})$ and the median of the variances.

The maximum likelihood estimators for location parameters associated with the normal and schizophrenic groups under the complex Gaussian model, are the following:

For the normal group:

$$\widetilde{\boldsymbol{\mu}}=(-1.5312+i0.6468,-0.0084+i0.4637,1.4306+i0.9779,1.2459-i2.1091,0.3962-i0.7200,$$
$$-0.0475-i1.3257,-0.2321-i1.9199,-0.6937-i0.2605,-0.6556-i1.0118,-1.3664+i0.1697,$$
$$-2.4834+i1.4590,0.2667+i0.0804)',$$

and $\widetilde{\sigma}^2 = 0.0159$ and for the schizophrenic group:

$$\widetilde{\boldsymbol{\mu}} = (-1.6898 + i0.0545, 0.0179 + i0.5385, 1.1107 + i1.3683, 1.7652 - i1.7264, 0.5794 - i0.6484, \\ 0.3610 - i1.300, 0.3015 - i1.9547, -0.6052 - i0.4795, -0.3432 - i1.2216, -1.3368 - i0.2235, \\ -2.8060 + i0.6043, 0.1087 - i0.0434)'$$

and $\widetilde{\sigma}^2 = 0.0155$.

Finally, we can test equality in affine shape between the two independent populations. In this experiment we have: two independent samples of 14 patients and 24 population shape parameters to estimate for each group. Namely, if $L(\boldsymbol{\mu}_n, \boldsymbol{\mu}_s)$ is the likelihood, where $\boldsymbol{\mu}_n$, $\boldsymbol{\mu}_s$, represent the mean shape parameters of the normal and schizophrenic group, respectively, then we want to test: $H_0 : \boldsymbol{\mu}_n = \boldsymbol{\mu}_s$ vs $H_1 : \boldsymbol{\mu}_n \neq \boldsymbol{\mu}_s$. Then $-2\log\Lambda = 2\sup_{H_1}\log L(\boldsymbol{\mu}_n, \boldsymbol{\mu}_s) - 2\sup_{H_0}\log L(\boldsymbol{\mu}_n, \boldsymbol{\mu}_s)$, and according to Wilk's theorem $-2\log\Lambda \sim \chi^2_{24}$ under H_0.

In this case we obtained that:

$$-2\log\Lambda = 2(858.09) - 2(834.60) = 46.98,$$

Since the p-value for the test is

$$P(\chi^2_{24} \geq 46.98) = 0.0034$$

we have important evidence that the normal and schizophrenic brain MR are different in affine shape. This conclusion is ratified by Dryden and Mardia (1998), for example, but using a "rigid" Euclidian match in $\Re^2$, in this case they obtained a p-value of 0.005.

Some new models (Kotz, Pearson VII, Bessel, Jensen-Logistic) can be studied and contrast them with the Gaussian one via BIC modified criteria, however new extensions of the so termed generalised Kummer relations with Jack polynomials are required (a generalisation to real normed division algebras of Herz (1955)); this shall constitute part of a future work.

Finally, observe that the real dimension of real normed division algebras can be expressed as potentia of 2, $\beta = 2^n$ for $n = 0, 1, 2, 3$. On the other hand, as observed by Kabe (1984), the results obtained in this work can be extended to hypercomplex cases; that is, for complex, bicomplex, biquaternion and bioctonion (or sedenionic) algebras, which of course are not division algebras (except the complex algebra), but are Jordan algebras, and all their isomorphic algebras. Note, too, that hypercomplex algebras are obtained by replacing the real numbers with complex numbers in the construction of real normed division algebras. Thus, the results for hypercomplex algebras are obtained by simply replacing β with 2β in our results (we reiterate, as reported by Kabe (1984)). Alternatively, following Kabe (1984), we can conclude that, our results are true for '2^n-ions', $n = 0, 1, 2, 3, 4, 5$, emphasising that only for $n = 0, 1, 2, 3$ are the result algebras in fact real normed division algebras.

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